# Poincaré series of a toric variety

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***Abstract.***— *For an affine toric variety $X$ we compute the Poincaré series of the multi-index filtration defined by a finite number of monomial divisorial valuations on the ring $\mathcal{O}_{X,0}$. We give an alternative description of the Poincaré series as an integral with respect to the Euler characteristic over the projectivization of the space of germs $\mathcal{O}_{X,0}$.*
*In particular we study divisorial valuations on the ring $\mathcal{O}_{\mathbb{C}^d,0}$ that arise by considering toric constellations. We give an explicit formula for the Poincaré series and a nice geometric description. This generalizes an expression of the Poincaré series for curves and rational surface singularities.*

A. Campillo, F. Delgado and S.M. Gusein-Zade computed the Poincaré series of the multi-index filtration on the ring of germs of functions in two variables defined by orders of a function on the branches of a reducible curve singularity in [C,D,G-Z2] and [C,D,G-Z5]. For plane curves they showed that the Poincaré series coincides with the Alexander polynomial of the link of the singularity. An intuitive motivation for this phenomenon is still missing. Also for an arbitrary collection of plane divisorial valuations the Poincaré series has been studied, see [D,G-Z]. In [C,D,G-Z6] the Poincaré series of the multi-index filtration on the ring of germs of functions on a rational surface singularity defined by the multiplicities of a function along components of the exceptional divisor of a resolution of the singularity has been computed.

These Poincaré series can be written in several ways. They can be described by the fibers of the corresponding extended semigroups. They also have the shape of an integral with respect to the Euler characteristic over the projectivization of the space of germs of functions (see [C,D,G-Z4] and [C,D,G-Z6]). This notion is similar to the notion of motivic integration and has been introduced in [C,D,G-Z3]. Furthermore, one has a description at the level of the modification space.

As the following shows, for an arbitrary affine variety $X$ with a finite set of divisorial valuations on the ring of germs $\mathcal{O}_{X,o}$, with $o$ a point on $X$, the multi-index filtration induced by the valuations does not always permit to define its Poincaré series.
Let $k$ be an arbitrary field. The Poincaré series of $X$ is given by a multi-index filtration with index set $\mathbb{Z}^r$ that is induced by a set of discrete valuations $\{\nu_1, \cdots, \nu_r\}$ of $\mathcal{O}_{X,o}$. For $\underline{v} = (v_1, \cdots, v_r) \in \mathbb{Z}^r$, such valuations induce complete ideals $I(\underline{v}) := \{g \in \mathcal{O}_{X,o} \mid \nu_i(g) \geq v_i, 1 \leq i \leq r\}$. When the centre of each valuation $\nu_i$, i.e. the set $\{g \in \mathcal{O}_{X,o} \mid \nu_i(g) > 0\}$, is the maximal ideal $\mathsf{m}$ of $\mathcal{O}_{X,o}$

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$(1 \leq i \leq r)$, the $k$-vector spaces $I(\underline{v})/I(\underline{v}+\underline{1})$ have finite dimension. Let us denote $d(\underline{v}) := \dim I(\underline{v})/I(\underline{v}+\underline{1})$. Then the series $L(t_1, \cdots, t_r) = \sum_{\underline{v}\in\mathbb{Z}^r} d(\underline{v})\underline{t}^{\underline{v}}$ is a well-defined Laurent series. Let $\underline{v}$ and $\underline{v}'$ be vectors in $\mathbb{Z}^r$, let $j \in \bar{J} := \{1, \cdots, r\}$ and suppose that $v_i = v'_i$ for all $i \in J\setminus j$. If $v_j \leq 0$ and $v'_j \leq 0$, then $I(\underline{v}) = I(\underline{v}')$ and hence $\prod_{i=1}^r (t_i - 1)L(t_1, \cdots, t_r) \in \mathbb{Z}[[t_1, \cdots, t_r]]$. The Poincaré series is then defined as

$$P(t_1, \cdots, t_r) = \frac{\prod_{i=1}^r (t_i - 1)L(t_1, \cdots, t_r)}{(t_1 \cdots t_r - 1)}.$$

However, if one of the valuations $\nu_1, \cdots, \nu_r$ does not have its centre at $\mathsf{m}$, then one can not define the series $L$. Indeed, suppose $\nu_i$ $(1 \leq i \leq r)$ is a valuation with centre at the prime ideal $\mathsf{p}_i$ which is different from $\mathsf{m}$. If $g \in \mathcal{O}_{X,o}$ and $\underline{\nu}(g) = \underline{v}$, then choose a function $h \in \mathsf{m} \setminus \mathsf{p}_i$. Now for each $n \in \mathbb{Z}_{\geq 0}$ one has that $gh^n \in I(\underline{v})$ and $gh^n \notin I(\underline{v}+\underline{1})$ and, as all the $gh^n (n \in \mathbb{Z}_{\geq 0})$ are linearly independent over $k$, it follows that $d(\underline{v})$ is infinite.

In this article we work over the field $k = \mathbb{C}$. We compute the Poincaré series for affine toric varieties and provide some equivalent descriptions for the Poincaré series such as exist in the cases of curves, rational surface singularities and plane divisorial valuations. As an interesting example, we study the case of divisorial valuations on the ring $\mathcal{O}_{\mathbb{C}^d,o} (d \geq 2)$ that are created by toric constellations.

## 1. Poincaré series of an affine toric variety

Let $\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ be a rational finite polyhedral strongly convex $d$-dimensional cone, where $N = \mathbb{Z}^d$. Let $M$ be the dual space to $N$, then there is a natural bilinear map $M \times N \to \mathbb{Z} : (m, n) \mapsto \langle m, n\rangle$. The dual cone $\check{\sigma}$ to $\sigma$ is defined as the set $\{m \in \mathbb{R}^d \mid \langle m, x\rangle \geq 0, \forall x \in \sigma\}$. When $S$ is a semigroup in $\check{\sigma} \cap M$ that generates $\check{\sigma}$ as cone, then also $S$ is finitely generated. Without loss of generality, we assume that $M$ is also the group generated by the semigroup $S$. We consider $\mathbb{C}[S]$ as the $S$-graded algebra $\oplus_{s\in S}\mathbb{C}\chi^s$. Let $X$ be the affine toric variety Spec$(\mathbb{C}[S])$. Note that $X$ is a normal variety if and only if $S = \check{\sigma} \cap M$. Let $\pi : X' \to X$ be an equivariant proper birational morphism, $X'$ being another toric variety. An irreducible codimension 1 subvariety $D$ of $X'$ induces a discrete valuation $\nu_D$ of $\mathbb{C}(X)$ and so an element $n_D$ of $N$ that acts as follows:

$$\begin{array}{rcl} n_D : M & \longrightarrow & \mathbb{Z} \\ m & \longmapsto & \nu_D(\chi^m). \end{array}$$

If $D$ is an irreducible component of $X' \setminus T$ then $n_D$ is a primitive element in $\sigma \cap N$. Vice versa a primitive element $n$ in $\sigma \cap N$ defines a discrete valuation $\nu$ of $\mathbb{C}(X)$ by setting $\nu(\sum_{m\in F} a_m\chi^m) = min\{n(m) \mid m \in F, a_m \neq 0\}$. In fact, one has $\nu = \nu_D$ for some $D$ in $X'$ for an appropriated $\pi : X' \to X$. Such valuations are usually called monomial valuations.
As we saw before, the valuations inducing the Poincaré series must have the

unique 0-dimensional orbit of $X$ as centre. If $D$ is an irreducible component of codimension 1 of $X' \setminus T$, then $\pi(D)$ is the closure of the orbit that is associated to the unique face $\tau$ of $\sigma$ such that $\overset{\circ}{\tau}$ contains $n_D$. This means that the valuations that we are considering correspond bijectively to the primitive elements of $\overset{\circ}{\sigma} \cap N$. In what follows we identify both notions. The above correspondence also stands for non-primitive elements in $\overset{\circ}{\sigma} \cap N$ and non-normalized monomial valuations.

We compute the Poincaré series $P_{X,H}$ of the affine toric variety $X$ with respect to an arbitrary set of divisorial monomial valuations $H = \{\nu_1, \cdots, \nu_r\}$ such that $H \subset \overset{\circ}{\sigma}$. In what follows $\underline{\nu}$ stands for the vector $(\nu_1, \cdots, \nu_r)$ and by $\langle s, \underline{\nu} \rangle$ we mean the vector $(\langle s, \nu_1 \rangle, \langle s, \nu_2 \rangle, \cdots, \langle s, \nu_r \rangle)$. We define the monomial cone $C$ as the cone generated by the values that can be obtained by valuating monomials, i.e. the cone generated by the set $\{\underline{\nu}(m) \mid \text{m monomial in } \mathbb{C}[S]\}$. One has a $\mathbb{Z}$-linear map $\phi : M \longrightarrow \mathbb{Z}^r$ given by $\phi(m) = \langle m, \underline{\nu} \rangle$. This map induces the following map among Laurent series groups

$$\begin{array}{rcl} \Phi : \mathbb{Z}[[M]] = \mathbb{Z}[[u_1, \cdots, u_d, u_1^{-1}, \cdots, u_d^{-1}]] & \longrightarrow & \mathbb{Z}[[\mathbb{Z}^r]] = \mathbb{Z}[[t_1, \cdots, t_r, t_1^{-1}, \cdots, t_r^{-1}]] \\ \sum_i \lambda_i \underline{u}^m & \longmapsto & \sum_i \lambda_i \underline{t}^{\langle m, \underline{\nu} \rangle}. \end{array}$$

Notice that for each commutative group $\Gamma$ the Laurent series group $\mathbb{Z}[[\Gamma]]$ is in fact a $\mathbb{Z}[\Gamma]$-module, and that the assignment of this module to $\Gamma$ is functorial. In particular, for our given semigroup $S$, one has the multi-graded Poincaré series of commutative algebra $Q(\underline{u}) = \sum_{s \in S} \underline{u}^s$ which is an element of $\mathbb{Z}[[S]]$ and so can be interpreted as an element in $\mathbb{Z}[[M]]$. In fact, this multi-graded Poincaré series is usually expressed as a rational function having $Q(\underline{u})$ as power series expansion. The Poincaré series in geometry $P_{X,H}(\underline{t})$ is an element in $\mathbb{Z}[[\mathbb{N}^r]]$ and so also an element of $\mathbb{Z}[[\mathbb{Z}^r]]$. The following result shows the relationship between both Poincaré series.

**Theorem 1.—** *The Poincaré series $P_{X,H}$ defined by the multi-index filtration induced by $H$ and associated to the multi-graduation of $S$, is the image under $\Phi$ of the Poincaré series of commutative algebra of the semigroup $S$, i.e.*

$$P_{X,H}(\underline{t}) = \Phi(Q(\underline{u})).$$

*Proof.* For a set $A \subset \{i_1, \cdots, i_s\}$, let $\alpha_A$ be the function

$$\begin{array}{rcl} \alpha_A : \mathbb{Z}^s & \longrightarrow & \mathbb{Z}^s \\ \underline{v} & \longmapsto & \underline{v}' \end{array}$$

where $v'_i = v_i - 1$ if $i \in A$ and $v'_i = v_i$ if $i \notin A$.
Then, for $\underline{v} \in \mathbb{Z}^r$, the coefficient of $\underline{t}^{\underline{v}}$ in $\prod_{i=1}^r (t_i - 1) L(t_1, \cdots, t_r)$ is

$$(-1)^r \sum_{A \subset \{1, \cdots, r\}} (-1)^{\# A} \dim \frac{I(\alpha_A(\underline{v}))}{I(\alpha_A(\underline{v}) + \underline{1})}$$

and the coefficient of $\underline{t}^{\underline{v}}$ in $P_{X,H}(\underline{t})$ is

$$\begin{array}{l} (-1)^{r+1} \displaystyle\sum_{A\subset\{1,\cdots,r\}} (-1)^{\#A} \dim \frac{I(\alpha_A(\underline{v}))}{I(\alpha_A(\underline{v})+\underline{1})} \quad + \\ (-1)^{r+1} \displaystyle\sum_{A\subset\{1,\cdots,r\}} (-1)^{\#A} \dim \frac{I(\alpha_A(\underline{v})-\underline{1})}{I(\alpha_A(\underline{v}))} \quad + \\ (-1)^{r+1} \displaystyle\sum_{A\subset\{1,\cdots,r\}} (-1)^{\#A} \dim \frac{I(\alpha_A(\underline{v})-\underline{2})}{I(\alpha_A(\underline{v})-\underline{1})} \quad + \quad \cdots \end{array}$$

This finite sum can be rewritten as

$$(-1)^{r+1} \sum_{A\subset\{1,\cdots,r\}} (-1)^{\#A} \dim \frac{\mathbb{C}[S]}{I(\alpha_A(\underline{v})+\underline{1})}.$$

Every subset $A \subset \{1,\cdots,r\}$ can be written in a unique way as $A_1 \times A_2$, with $A_1 \subset \{2,\cdots,r\}$ and $A_2 \subset \{1\}$. We group the terms having the same component $A_1$ and we get

$$\begin{array}{rl} & \displaystyle\sum_{A\subset\{1,\cdots,r\}} (-1)^{\#A} \dim \frac{\mathbb{C}[S]}{I(\alpha_A(\underline{v})+\underline{1})} \\ = & \displaystyle\sum_{A\subset\{2,\cdots,r\}} (-1)^{\#A} \#\{\chi^s \mid \langle s,\nu_1\rangle = v_1, \langle s,(\nu_2,\cdots,\nu_r)\rangle \geq \alpha_A(v_2,\cdots,v_r)+\underline{1}\}. \end{array}$$

We go on simplifying in the same way, so now we write every subset $A \subset \{2,\cdots,r\}$ as $A_1 \times A_2$, with $A_1 \subset \{3,\cdots,r\}$ and $A_2 \subset \{2\}$ and so on. At each step we group the terms having the same component $A_1$ and we obtain

$$\begin{array}{rl} & \displaystyle\sum_{A\subset\{3,\cdots,r\}} (-1)^{\#A+1} \#\{\chi^s \mid \langle s,\nu_1\rangle = v_1, \langle s,\nu_2\rangle = v_2, \\ & \qquad\qquad \langle s,(\nu_3,\cdots,\nu_r)\rangle \geq \alpha_A(v_3,\cdots,v_r)+\underline{1}\} \\ = & \displaystyle\sum_{A\subset\{4,\cdots,r\}} (-1)^{\#A+2} \#\{\chi^s \mid \langle s,\nu_1\rangle = v_1, \langle s,\nu_2\rangle = v_2, \langle s,\nu_3\rangle = v_3, \\ & \qquad\qquad \langle s,(\nu_4,\cdots,\nu_r)\rangle \geq \alpha_A(v_4,\cdots,v_r)+\underline{1}\} \\ \vdots & \\ = & (-1)^{r-1} \#\{\chi^s \mid \langle s,\underline{\nu}\rangle = \underline{v}\}. \end{array}$$

Hence the Poincaré series $P_{X,H}(\underline{t})$ is

$$\sum_{\underline{v}\in\mathbb{Z}^r} \#\{\chi^s \mid \langle s,\underline{\nu}\rangle = \underline{v}\} \underline{t}^{\underline{v}}.$$

∎

In what follows we will write $N(\underline{v})$ when we want to refer to the number $\#\{s \in S \mid \langle s, \underline{\nu} \rangle = \underline{v}\}$. Since $\sigma$ is $d$-dimensional and strongly convex, one has that $N(\underline{v})$ is finite for every $\underline{v}$.

The semigroup of $X$ with respect to the set of valuations $H$ is

$$S_H = \{\underline{v} \in \mathbb{Z}^r \mid \underline{v} = \underline{\nu}(g) \text{ for some } g \in \mathcal{O}_{X,o}\}.$$

Let $g$ be a function in $I(\underline{v})$, say $g = \sum_{i=1}^{s} \lambda_i m_i$ where the $m_i$ are monomials and the $\lambda_i$ are complex numbers ($1 \leq i \leq s$). For $j \in J$, we denote by $a_j(g)$ the part $\sum_{i \in K_j} \lambda_i m_i$, where $K_j = \{i \mid \nu_j(m_i) = v_j\}$, and by $\underline{a}(g) = (a_1(g), \cdots, a_r(g))$. The set $\hat{S}_H = \{(\underline{\nu}(g), \underline{a}(g)) \mid g \in \mathcal{O}_{X,o}\}$ is a semigroup with respect to the summation of the components $\underline{\nu}$ and multiplication of the parts $\underline{a}$ and is called the extended semigroup. This notion showed up for the first time in [C,D,G-Z1] where it was introduced for plane curves. Later this notion has been extended in the study of the Poincaré series of plane divisorial valuations and rational surface singularities.
For $j \in J$, denote by $D_j(\underline{v})$ the complex vector space $I(\underline{v})/I(\underline{v} + \underline{e_j})$ where $\underline{e_j}$ is the $r$-tuple with $j$-th component equal to 1 and the other components equal to 0. Consider the application

$$\begin{array}{rcl} j_{\underline{v}} : I(\underline{v}) & \longrightarrow & D_1(\underline{v}) \times \cdots \times D_r(\underline{v}) \\ g & \longmapsto & (a_1(g), \cdots, a_r(g)) = \underline{a}(g). \end{array}$$

Let $D(\underline{v})$ be the image of the map $j_{\underline{v}}$. Then $D(\underline{v}) \simeq I(\underline{v})/I(\underline{v}+\underline{1})$ and we define $F_{\underline{v}}$ as $D(\underline{v}) \cap (D_1^*(\underline{v}) \times \cdots \times D_r^*(\underline{v}))$ where $D_j^*(\underline{v})$ denotes $D_j(\underline{v}) \setminus \{\underline{0}\}$, $j \in J$. Having the map

$$\begin{array}{rrcl} \rho : & \hat{S}_H & \longrightarrow & S_H \\ & (\underline{\nu}(g), \underline{a}(g)) & \longmapsto & \underline{\nu}(g), \end{array}$$

$F_{\underline{v}}$ can also be expressed as $\rho^{-1}(\underline{v})$ and therefore one also calls the spaces $F_{\underline{v}}$ the fibers of the extended semigroup $\hat{S}_H$. For $\underline{v} \in S_H$, the space $F_{\underline{v}}$ is the complement to an arrangement of vector subspaces in the vector space $D(\underline{v})$. Moreover $F_{\underline{v}}$ is invariant with respect to multiplication by nonzero constants. Let $\mathbb{P}F_{\underline{v}} = F_{\underline{v}}/\mathbb{C}^*$ be the projectivization of $F_{\underline{v}}$. It is the complement to an arrangement of projective subspaces in the projective space $\mathbb{P}D(\underline{v})$.
As the ideal $I(\underline{v})$ is a monomial ideal, one can see the space $F_{\underline{v}}$ as the set of functions $g = \sum_{i=1}^{s} \lambda_i m_i$ in $\mathcal{O}_{X,o}$ (the $m_i$ are monomials and the $\lambda_i, 1 \leq i \leq s$, are complex numbers different from 0) for which $\underline{\nu}(g) = \underline{v}$ and for which for all $i \in \{1, \cdots, s\}$ holds that $m_i \in I(\underline{v})$ and that there exists a $j \in J$ such that $\nu_j(m_i) = \nu_j(g)$. If a function $g$ with $\underline{\nu}(g) = \underline{v}$ has this form, we say that $g$ is in reduced form. We write $\mathrm{supp}(g)$ for the support of $g$, which is the set $\{m_i \mid 1 \leq i \leq s\}$.
Now let us fix $\underline{v} \in \mathbb{Z}^r$. Let $\mathcal{M}$ be the set of all monomials that can appear in the support of some $g$ in $\mathbb{P}F_{\underline{v}}$, so $\mathcal{M} = \{m \text{ monomial} \mid m \in I(\underline{v}) \text{ and } \exists j \in J : \nu_j(m) = v_j\}$. Note that $\mathcal{M}$ is a finite set. For a subset $L$ of $\mathcal{M}$, let $\underline{\nu}(L)$ be the vector $\underline{w} \in \mathbb{Z}^r$ with $w_j = \min\{\nu_j(m) \mid m \in L\}$, $j \in J$.

**Proposition 1.—**

$$P(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}^r} \chi(\mathbb{P}F_{\underline{v}}) \underline{t}^{\underline{v}}.$$

*Proof.*
We write $\mathbb{P}F_{\underline{v}}$ as a disjoint union:

$$\mathbb{P}F_{\underline{v}} = \bigcup_{L \subset \mathcal{M}, \underline{\nu}(L) = \underline{v}} \{g \in \mathbb{P}F_{\underline{v}} \mid \mathrm{supp}(g) = L\}.$$

For $\Lambda_L = \{g \in \mathbb{P}F_{\underline{v}} \mid \mathrm{supp}(g) = L\}$, one has that $\chi(\Lambda_L) = \chi(\mathbb{C}^{*k}) = 0$ for some $k \in \mathbb{Z}_{>0}$ when $L$ is not a singleton, and $\chi(\Lambda_L) = 1$ when $L$ is a singleton. This gives us

$$\chi(\mathbb{P}F_{\underline{v}}) = N(\underline{v}).$$

■

This result can also be obtained by the combinatorial proof given in [C,D,G-Z2] or in [C,D,G-Z3]. Actually their proof works for every variety for which the Poincaré series exists. Exploiting the fact that our varieties are toric, we obtain this explicit and much shorter proof.

Analogously to the case of curves and rational surface singularities, we write the Poincaré series as an integral with respect to the Euler characteristic over the projectivization $\mathbb{P}\mathcal{O}_{X,o}$. We first recall this notion, introduced in [C,D,G-Z3] and inspired by the notion of motivic integration (see for example [D,L]). It was developed to integrate over $\mathbb{P}\mathcal{O}_{\mathbb{C}^n,o}$ what is not allowed by the usual Viro construction where one integrates with respect to the Euler characteristic over finite dimensional spaces (see [V]). It can be extended to integrals over $\mathbb{P}\mathcal{O}_{X,o}$, for $X$ an arbitrary variety.

Let $\mathsf{m}$ be the maximal ideal in the ring $\mathcal{O}_{X,o}$ and for $k \in \mathbb{Z}_{\geq 0}$, let $\mathcal{J}^k_{X,o} = \mathcal{O}_{X,o}/\mathsf{m}^{k+1}$ be the space of $k$-jets of functions on the toric variety $X$. For a complex vector space $L$ (finite of infinite dimensional) let $\mathbb{P}L = (L \setminus \{0\})/\mathbb{C}^*$ be its projectivization and let $\mathbb{P}^*L$ be the disjoint union of $\mathbb{P}L$ with a point. One has a natural map $\pi_k : \mathbb{P}\mathcal{O}_{X,o} \mapsto \mathbb{P}^*\mathcal{J}^k_{X,o}$.

**Definition.**— A subset $A \subset \mathbb{P}\mathcal{O}_{X,o}$ is said to be cylindrical if $A = \pi_k^{-1}(B)$ for a constructible (i.e. a finite union of locally closed sets) subset $B \subset \mathbb{P}\mathcal{J}^k_{X,o} \subset \mathbb{P}^*\mathcal{J}^k_{X,o}$.

**Definition.**— For a cylindrical subset $A \subset \mathbb{P}\mathcal{O}_{X,o}$ ($A = \pi_k^{-1}(B)$, $B \subset \mathbb{P}\mathcal{J}^k_{X,o}$), its Euler characteristic $\chi(A)$ is defined as the Euler characteristic $\chi(B)$ of the set $B$.

Let $\psi : \mathbb{P}\mathcal{O}_{X,o} \to G$ be a function which takes values in an abelian group $G$.

**Definition.**— We say that the function $\psi$ is cylindrical if, for each $g \in G, g \neq 0$, the set $\psi^{-1}(g) \subset \mathbb{P}\mathcal{O}_{X,o}$ is cylindrical.

**Definition.—** The integral of a cylindrical function $\psi$ over the space $\mathbb{P}\mathcal{O}_{X,o}$ with respect to the Euler characteristic is

$$\int_{\mathbb{P}\mathcal{O}_{X,o}} \psi d\chi := \sum_{g\in G, g\neq 0} \chi(\psi^{-1}(g)) \cdot g$$

if this sum makes sense in $G$. If the integral exists, the function $\psi$ is said to be integrable.

Let $\mathbb{Z}[[\underline{t}]] = \mathbb{Z}[[t_1, \cdots, t_r]]$ be the group with respect to the addition of formal power series in the variables $t_1, \cdots, t_r$. We have the map $\underline{\nu} : \mathbb{P}\mathcal{O}_{X,o} \longrightarrow \mathbb{Z}^r$. Let $\underline{t}^{\underline{\nu}}$ be the corresponding function with values in $\mathbb{Z}[[\underline{t}]]$.

**Proposition 2.—**

$$P(\underline{t}) = \textstyle\int_{\mathbb{P}\mathcal{O}_{X,o}} \underline{t}^{\underline{\nu}} d\chi.$$

*Proof.* For $\underline{v} \in \mathbb{Z}^r_{\geq 0}$, let $N = 1+ \max\{v_i \mid 1 \leq i \leq r\}$ and let $Y_{\underline{v}} = \{j^N g \in \mathbb{P}\mathcal{J}^N_{X,o} \mid g \in \mathbb{P}F_{\underline{v}}\} \subset \mathbb{P}\mathcal{J}^N_{X,o}$. Then $\{g \in \mathbb{P}\mathcal{O}_{X,o} \mid \underline{\nu}(g) = \underline{v}\} = \pi_N^{-1}(Y_{\underline{v}})$. Consider the map

$$\begin{array}{rcl} \alpha : Y_{\underline{v}} & \longrightarrow & F_{\underline{v}} \\ j^N g & \longmapsto & \underline{a}(g). \end{array}$$

This map is $\mathbb{C}^*$-invariant and so can be considered as a map $\alpha : Y_{\underline{v}} \longrightarrow \mathbb{P}F_{\underline{v}}$. As $\alpha$ is a locally trivial fibration whose fibre is a complex affine space we obtain

$$\int_{\mathbb{P}\mathcal{O}_{X,o}} \underline{t}^{\underline{\nu}} d\chi = \sum_{\underline{v}\in\mathbb{Z}^r} \chi(Y_{\underline{v}}) \underline{t}^{\underline{v}} = \sum_{\underline{v}\in\mathbb{Z}^r} \chi(\mathbb{P}F_{\underline{v}}) \underline{t}^{\underline{v}} = P(\underline{t}).$$

■

## 2. Poincaré series of $\mathbb{C}^d$ induced by a toric constellation

In this section we look at the particular case where $X$ is $\mathbb{C}^d$ endowed with the action of the torus $T \cong (\mathbb{C}^*)^d$ $(d \geq 2)$. We study the Poincaré series of $\mathbb{C}^d$ where the modification $\pi$ is given by a toric constellation $\mathcal{C}$ with origin. Let us first recall the basic notions in the context of toric constellations (see for example [C,G-S,L-J]).

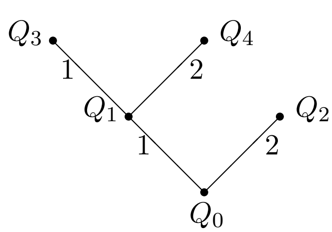
Let $Z$ be a variety obtained from $X$ by a finite succession of point blowing-ups. A point $Q \in Z$ is said to be infinitely near to a point $O \in X$ if $O$ is in the image of $Q$; we write $Q \geq O$. A constellation is a finite sequence $\mathcal{C} = \{Q_0, Q_1, \cdots, Q_{r-1}\}$ of infinitely near points of $X$ such that $Q_0 \in X = X_0$ and each $Q_j$ is a point on the variety $X_j$ obtained by blowing up $Q_{j-1}$ in $X_{j-1}$, $j \in J$. When $Q_j$ is a 0-dimensional $T$-orbit in the toric variety $X_j$, for $j \in J$, the constellation is said to be toric.

The relation '$\geq$' gives rise to a partial ordering on the points of a constellation. In the case that they are totally ordered, so $Q_r \geq \cdots \geq Q_0$, the constellation $\mathcal{C}$ is called a chain.

For every $Q_j$ in $\mathcal{C}$, the subsequence $C^j = \{Q_i \mid Q_j \geq Q_i\}$ of $\mathcal{C}$ is a chain. The integer $l(Q_j) = \#\mathcal{C}^j - 1$ is called the level of $Q_j$. In particular $Q_0$ has level 0. If no other point of $\mathcal{C}$ has level 0 then $Q_0$ is called the origin of $\mathcal{C}$.
A tree with a root such that each vertex has at most $d$ following adjacent vertices is called a $d$-ary tree. There is a natural bijection between the set of $d$-dimensional toric constellations with origin and the set of finite $d$-ary trees with a root, with the edges weighted with positive integers not greater than $d$, such that two edges with the same source have different weights. We simply say 'weights' when we want to refer to the weights on the edges. The weights indicate in which affine chart the points of the constellation are created (see [C,G-S,L-J] for details).

Let $E_j$ $(j \in J)$ be the irreducible components of the exceptional divisor $\mathcal{D}$ created by blowing up the constellation $\mathcal{C} = \{Q_0, Q_1, \cdots, Q_{r-1}\}$ and let $\xi_j$ be the generic point of $E_j$ $(j \in J)$. Then $\mathcal{O}_{X,\xi_j}$ is a discrete valuation ring. We denote the induced valuation by $\nu_j$.
We denote the matrix of the linear system of equations $\langle s, \underline{\nu} \rangle = \underline{v}$ by $\mathcal{L}(\mathcal{C})$ and we denote the column vectors of $\mathcal{L}(\mathcal{C})$ by $\underline{v}_1, \cdots, \underline{v}_d$. Let $C$ be the cone in $\mathbb{R}^r_{\geq 0}$ generated by $\underline{v}_1, \cdots, \underline{v}_d$. Note that $C$ is the monomial cone associated to the constellation $\mathcal{C}$. Recall that the cone is regular if it can be generated by a part of a basis of $\mathbb{Z}^r$. If $\mathcal{L}(\mathcal{C})$ has rank $s$ and if $s < d$, then the cone is said to be degenerated.

**Example:**




Suppose $d = 4$ and $\mathcal{C}$ is the constellation pictured at the left. By blowing up in the origin $Q_0$ we get an exceptional variety $B_0 \cong \mathbb{P}^3$. In $B_0$ there are 2 points in which we blow up, namely $Q_1$ and $Q_2$. For example the point $Q_1$ is the origin of the affine chart induced by the weight 1, we shortly denote this affine chart by '1'.
After blowing up in $Q_1$ we get an exceptional variety $B_1 \cong \mathbb{P}^3$, where again we blow up in 2 points. The point $Q_3$ is the origin of the affine chart '$1-1$' and $Q_4$ is the origin of the affine chart '$1-2$'. The associated linear system $\mathcal{L}(\mathcal{C})$ is

$$\left\{\begin{array}{ccccccccc} a & + & b & + & c & + & d & = & v_1 \\ a & + & 2b & + & 2c & + & 2d & = & v_2 \\ 2a & + & b & + & 2c & + & 2d & = & v_3 \\ a & + & 3b & + & 3c & + & 3d & = & v_4 \\ 2a & + & 3b & + & 4c & + & 4d & = & v_5 \end{array}\right.$$

and $C$ is the cone $\langle (1,1,2,1,2), (1,2,1,3,3), (1,2,2,3,4) \rangle \subset \mathbb{R}^5_{\geq 0}$ which is obviously degenerated.

$\square$

To know the Poincaré series one can use Theorem 1. Let $\underline{v}_1, \cdots, \underline{v}_d$ be the column vectors of $\mathcal{L}(\mathcal{C})$. Then it follows by Theorem 1 that

$$P(\underline{t}) = \frac{1}{(1-\underline{t}^{\underline{v}_1})\cdots(1-\underline{t}^{\underline{v}_d})}.$$

One can also obtain $P(\underline{t})$ by computing the numbers $N(\underline{v}) = \#\{s \in \mathbb{N}^d \mid \langle s, \underline{\nu}\rangle = \underline{v}\}$ for each $\underline{v} \in \mathbb{Z}^r$. We determine them as later these values will be useful for us. The following proposition gives some properties of the cone $C$ which will allow us to compute the numbers $N(\underline{v})$.

**Proposition 3.—**

1. *$C$ is a regular cone;*

2. *$C$ is degenerated if and only if the number of different weights appearing in the constellation is less than or equal to $d-2$.*

*Proof.* If the number of different weights appearing in the constellation $\mathcal{C}$ is less than or equal to $d-2$, then at least 2 columns in $\mathcal{L}(\mathcal{C})$ are equal and $C$ is degenerated. Suppose that the number of different weights is bigger than $d-2$, say that $1, \cdots, d-1$ are weights appearing in the constellation and let $Q_1, \cdots, Q_{d-1}$ be points in the constellation such that $Q_i$ is a point with minimal level arising in an affine chart induced by the weight $i$ and such that whenever $Q_i \geq Q_j$ and $Q_i \neq Q_j$, then $i > j$ $(1 \leq i, j \leq d-1)$.
The linear equations induced by the origin $Q_0$ and by $Q_1, \cdots, Q_{d-1}$ give rise to a linear system whose determinant can be supposed to be of the form

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 2 & 2 & \cdots & \cdots & 2 \\ * & a_2-1 & a_2 & \cdots & \cdots & a_2 \\ * & * & a_3-1 & a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & \cdots & * & a_{d-1}-1 & a_{d-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ -1 & 0 & 0 & \cdots & \cdots & 0 \\ * & -1 & 0 & \cdots & \cdots & 0 \\ * & * & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & \cdots & * & -1 & 0 \end{vmatrix},$$

where $a_2, \cdots, a_{d-1}$ are integer numbers. As this determinant is equal to 1, the cone $C$ is regular and non-degenerated.
Now if $C$ is degenerated, and generated by $s = \operatorname{rank}(\mathcal{L}(\mathcal{C}))$ vectors then there are $s-1$ different weights appearing in the constellation. In the same way as above we obtain a $(s \times s)$-determinant which is equal to 1 such that also in this case $C$ is regular.

■

Note that it follows from the proof that saying that the cone $C$ is degenerated, is the same as saying that there is one column that appears at least twice in $\mathcal{L}(\mathcal{C})$ and that there are no other linear dependencies between the columns of $\mathcal{L}(\mathcal{C})$.

- If $C$ is non-degenerated then $N(\underline{v}) = 1$ if $\underline{v} \in C$, else $N(\underline{v}) = 0$. Suppose that $\underline{v}_1, \cdots, \underline{v}_d$ are the column vectors of the linear system $\mathcal{L}(\mathcal{C})$. As $C$ is regular, we obtain again
$$P(\underline{t}) = \frac{1}{(1-\underline{t}^{\underline{v}_1}) \cdots (1-\underline{t}^{\underline{v}_d})}.$$
- When $C$ is degenerated, let then $\underline{v}_1, \cdots, \underline{v}_s$ be the $s$ different vectors that generate $C$ and let $\underline{v}_s$ be the vector that appears at least twice as column vector in $\mathcal{L}(\mathcal{C})$. As $C$ is regular, one can write each $\underline{v} \in C$ in a unique way as $\underline{v} = \lambda_1 \underline{v}_1 + \cdots + \lambda_{s-1}\underline{v}_{s-1} + \lambda \underline{v}_s$, for some $\lambda_i, \lambda \in \mathbb{Z}_{\geq 0}$ $(1 \leq i \leq s-1)$. Setting $k = d - s + 1$, a simple calculation shows that
$$N(\underline{v}) = \binom{k+\lambda-1}{\lambda}$$
if $\underline{v} \in C$, else $N(\underline{v}) = 0$. Using that $\sum_{\lambda \in \mathbb{Z}_{\geq 0}} \binom{k+\lambda-1}{\lambda} x^\lambda = \frac{1}{(1-x)^k}$, one sees again that
$$P(\underline{t}) = \frac{1}{(1-\underline{t}^{\underline{v}_1}) \cdots (1-\underline{t}^{\underline{v}_{s-1}})(1-\underline{t}^{\underline{v}_s})^k}.$$

For the example given above the Poincaré series is then
$$\frac{1}{(1-t_1t_2t_3^2t_4t_5^2)(1-t_1t_2^2t_3t_4^3t_5^3)(1-t_1t_2^2t_3^2t_4^3t_5^4)^2}.$$

For curves, rational surface singularities and plane divisorial valuations, there exists a description of the Poincaré series at the level of the modification space. Let $\mathcal{D} = \bigcup_{j=1}^r E_j$ be the exceptional variety with irreducible components $E_j, j \in J$. We denote by $\overset{\circ}{E}_j$ the smooth part of the irreducible component $E_j$, i.e. without intersection points with all other components of the exceptional divisor. Let $M = -(E_i \circ E_j)$ be minus the intersection matrix of the components of the exceptional variety $\mathcal{D}$. Let $\nu_j$ be the discrete valuation on the local ring $\mathcal{O}_{X,o}$ induced by $E_j$. The semigroup of values $S := \{\underline{\nu}(g) \mid g \in \mathcal{O}_{X,o}\}$ is exactly the set of vectors $\{\underline{v} \in \mathbb{Z}^r_{\geq 0} \mid \underline{v}M \geq \underline{0}\}$. For a topological space $E$, let $S^nE = E^n/S_n$ $(n \geq 0)$ be the $n$-th symmetric power of the space $E$, i.e. the space of $n$-tuples of points of the space $E$ ($S^0E$ is a point).
Campillo, Delgado and Gusein-Zade construct the space
$$Y = \bigcup_{\{\underline{v} \in S\}} \left( \prod_{j=1}^r S^{n_j} \overset{\circ}{E}_j \right),$$
where $\underline{v}M = (n_1, \cdots, n_r) =: \underline{n}(\underline{v})$. For $g \in \mathcal{O}_{X,o}$, $g \neq 0$ and $\underline{v} = \underline{\nu}(g)$, the number $n_j(\underline{v})$ is equal to the intersection number of the strict transform

of $g$ with $E_j$. Let $Y_{\underline{v}}$ be the connected component $\prod_{j=1}^{r} S^{n_j} \overset{\circ}{E}_j$ of $Y$, where $(n_1, \cdots, n_r) = \underline{n}(\underline{v})$. They show that

$$P(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}^r} \chi(Y_{\underline{v}}) \underline{t}^{\underline{v}}. \tag{1}$$

In the case of curves and plane divisorial valuations, this description induces the elegant formula of the Poincaré series where the exponents can be written as the Euler characteristics of the smooth parts $\overset{\circ}{E}_j$.

In what follows we prove a generalized form of (1) for toric constellations. I want to thank Sabir Gusein-Zade for the very stimulating conversation about this subject.

For $j \in J$, let $B_j$ be the projective $(d-1)$-dimensional exceptional variety that is created by blowing up the point $Q_j$ in $X_j$. When we want to refer to the strict transform of $B_j$ at some intermediate stadium, we also write $E_j$. If $g \in \mathcal{O}_{\mathbb{C}^d, o}$, then we write $\hat{g}$ for the strict transform at the end of the process as well as for the strict transforms of $g$ in the intermediate stadia. For $v \in \mathbb{Z}^r$, we define the set

$$\begin{aligned} D_{\underline{v}} : \;=\; & \{\{\hat{g} = 0\} \cap \mathcal{D} \mid g \in \mathcal{O}_{\mathbb{C}^d, o},\ \underline{\nu}(g) = \underline{v} \text{ and } \{\hat{g} = 0\} \text{ does not contain} \\ & \text{any non-empty intersection } E_a \cap E_b, a, b \in J, a \neq b\}. \end{aligned}$$

Obviously to know $D_{\underline{v}}$ it is sufficient to consider the elements $g$ in $\mathbb{P}F_{\underline{v}}$. We make a topological space of it as follows. We write $E_{\underline{v}}$ for the sum $\sum_{i=1}^{r} v_i E_i$ and $M$ for the line bundle associated to $E_{\underline{v}}$. The restriction $R$ of $\mathcal{O}_Z(-E_{\underline{v}}) \otimes M^{-1}$ to $\mathcal{D}$ is a line bundle and as $\mathcal{D}$ is a projective variety, the global sections of $R$ form a finite dimensional vector space. For $g \in F_{\underline{v}}$, the divisor $\hat{g} \cap \mathcal{D}$ is the divisor of zeroes of a global section of $R$. Then $D_{\underline{v}}$ can be seen as a subset of the projectivization of this vector space.

**Theorem 2.**— *The Poincaré series $P(\underline{t})$ is equal to*

$$\sum_{\underline{v} \in \mathbb{Z}^r} \chi(D_{\underline{v}}) \underline{t}^{\underline{v}}.$$

To achieve the result we will construct a subspace $Z_{\underline{v}}$ of $\mathbb{P}F_{\underline{v}}$ that has the same Euler characteristic as $\mathbb{P}F_{\underline{v}}$ and such that there exists a homeomorphism of $Z_{\underline{v}}$ with $D_{\underline{v}}$. Then Theorem 2 will follow by Proposition 1.

The obvious candidate for the space $Z_{\underline{v}}$ is the set

$$\begin{aligned} Z_{\underline{v}} \;:=\; & \{g \in \mathbb{P}F_{\underline{v}} \mid \{\hat{g} = 0\} \text{ does not contain any non-empty} \\ & \text{intersection } E_a \cap E_b, a, b \in J, a \neq b\}. \end{aligned}$$

**Lemma 1.—**

$$\chi(Z_{\underline{v}}) = \chi(\mathbb{P}F_{\underline{v}}).$$

*Proof.* Let $g = \sum_{i=1}^{s} \lambda_i m_i$ be in reduced form and suppose that $E_a \cap E_b \neq \emptyset$ for some $a$ and $b$ in $J$, $a \neq b$. If $g$ contains $E_a \cap E_b$ then also $g_{\underline{\mu}} = \sum_{i=1}^{s} \mu_i m_i$ contains $E_a \cap E_b$, for all $\mu = (\mu_1, \cdots, \mu_s) \in \mathbb{C}^{*s}$. This yields that

$$\begin{aligned}
\chi(Z_{\underline{v}}) &= \chi(\{g \in \mathbb{P}F_{\underline{v}} \mid g \text{ a monomial and } \{\hat{g} = 0\} \text{ does not contain any} \\
&\qquad \text{non-empty intersection } E_a \cap E_b, a, b \in J, a \neq b\}) \\
&= \chi(\{g \in \mathbb{P}F_{\underline{v}} \mid g \text{ a monomial}\}) \\
&= N(\underline{v}) \\
&= \chi(\mathbb{P}F_{\underline{v}}).
\end{aligned}$$

■

Now we investigate the map

$$\begin{aligned}
\phi : Z_{\underline{v}} &\longrightarrow D_{\underline{v}} \\
g &\longmapsto \{\hat{g} = 0\} \cap \mathcal{D}.
\end{aligned}$$

The following lemma tells us how $\{\hat{g} = 0\} \cap \mathcal{D}$ looks like.

**Lemma 2.—** *Consider $g \in \mathbb{P}F_{\underline{v}}$, a function in reduced form, say $g = \sum_{i=1}^{s} \lambda_i m_i$. For $j \in J$, let $\Lambda_{g,j}$ be the set $\{m \in \text{ supp}(g) \mid \nu_j(m) = v_j\}$. Then the equation of $\{\hat{g} = 0\} \cap E_j$ is $\sum_{m_i \in \Lambda_{g,j}} \lambda_i \hat{m}_i = 0$ in every affine chart where the intersection $\{\hat{g} = 0\} \cap E_j$ is visible.*

*Proof.* When $\{\hat{g} = 0\} \cap B_j \neq \emptyset$, then in an affine chart covering $B_j \cong \mathbb{P}^{d-1}$ it can be described by the equation $\sum_{i \in K_j} \lambda_i \hat{m}_i = 0$ where

$$\begin{aligned}
K_j &= \{i \mid \nu_j(m_i) = \min\{\nu_j(m) \mid m \in \text{supp}(g)\}\} \\
&= \{i \mid \nu_j(m_i) = v_j\} \\
&= \Lambda_{g,j}.
\end{aligned}$$

Now Lemma 2 follows directly.

■

Note that $\hat{m}$ can be equal to 1 in some affine chart covering $E_j$ $(j \in J)$. However, there exists always a $j \in J$ such that $\{\hat{m} = 0\} \cap E_j \neq \emptyset$.

Let us have a look at the following example. Lemma 2 allows us to deduce quickly a necessary condition on a subset $S$ of $\mathbb{P}F_{\underline{v}}$ for the sets $\{g \in S\}$ and $\{\{\hat{g} = 0\} \cap \mathcal{D} \mid g \in S\}$ to be in $1-1$ correspondence.

**Example (continued).—**

Let $\underline{v}$ be $(6, 11, 10, 14, 18)$. Take $g(x, y, z, u) = y^2z^4 + x^3y^4 + x^{14}$. The values of the monomials in the support of $g$ are:

$$\nu(y^2z^4) = (6, 12, 10, 18, 22), \quad \nu(x^3y^4) = (7, 11, 10, 15, 18) \quad \text{and}$$
$$\nu(x^{14}) = (14, 14, 28, 14, 28).$$

Lemma 2 tells us that the equation of $\hat{g} \cap E_1$ in an affine chart where the intersection is visible, is $\widehat{y^2z^4}$. For $\hat{g} \cap E_2$ it is $\widehat{x^3y^4}$, for $\hat{g} \cap E_3$ it is $\widehat{y^2z^4} + \widehat{x^3y^4}$, for $\hat{g} \cap E_4$ it is $\widehat{x^{14}}$ and for $\hat{g} \cap E_5$ it is $\widehat{x^3y^4}$. It follows that the strict transform of $h(x, y, z, u) = y^2z^4 + x^3y^4 + \mu x^{14}$, with $\mu \neq 0$, has the same intersection with $\mathcal{D}$ as $\hat{g}$.

$\square$

We formalize what the example shows. As in Proposition 1, write $\mathbb{P}F_{\underline{v}}$ as the disjoint union $\bigcup_{L \subset \mathcal{M}, \underline{\nu}(L) = \underline{v}} \Lambda_L$. Let $L$ be a support appearing in this disjoint union. For $j \in J$, we define $\Lambda_{L,j}$ as the set of monomials $m$ in $L$ such that $\nu_j(m) = v_j$. The observation made in the example shows that the map

$$\begin{array}{rcl} \phi_L : \{g \in \mathbb{P}F_{\underline{v}} \mid \mathrm{supp}(g) = L\} & \longrightarrow & \{\{\hat{g} = 0\} \cap \mathcal{D} \mid g \in \mathbb{P}F_{\underline{v}} \text{ and } \mathrm{supp}(g) = L\} \\ g & \longmapsto & \{\hat{g} = 0\} \cap \mathcal{D} \end{array}$$

certainly can not be a bijection if there exists a subset $D$ of $J$, with the following properties:

$$\exists a, b \in J : a \in D, \quad b \notin D, \quad (\cup_{d \in D} \Lambda_{L,d}) \cap (\cup_{d \notin D} \Lambda_{L,d}) = \emptyset. \tag{2}$$

Indeed, if such a subset $D$ exists, then write $g = g_a + g_b$ where $\mathrm{supp}(g_a) = \cup_{d \in D} \Lambda_{L,d}$ and $\mathrm{supp}(g_b) = \cup_{d \notin D} \Lambda_{L,d}$. Then by Lemma 2 it follows that for $g_{\underline{\lambda}} = \lambda_a g_a + \lambda_b g_b$, the transforms $\{\widehat{g_{\underline{\lambda}}} = 0\}$ and $\{\hat{g} = 0\}$ have the same intersection with $\mathcal{D}$ for all $\underline{\lambda} = (\lambda_a, \lambda_b) \in \mathbb{C}^{*2}$. We claim that also the converse is true.

**Proposition 4.—** *Let $L \subset \mathcal{M}$ be as above. If for all $a, b \in J$, $a \neq b$, holds that there exists no subset $D$ with $a \in D$, $b \notin D$ and $(\cup_{d \in D} \Lambda_{L,d}) \cap (\cup_{d \notin D} \Lambda_{L,d}) = \emptyset$, then the map $\phi_L$ is a bijection.*

*Proof.*

1. First we show that for a monomial $m$ in $L$, when given $\{\hat{m} = 0\} \cap \mathcal{D}$, one can find $m$ again. So suppose $m$ in $L$. Take a $j \in J$ such that $\nu_j(m) = v_j$ and such that $\{\hat{m} = 0\} \cap E_j$ is visible in an affine chart in the final stadium, say in the chart presented as $c_1 - c_2 - \cdots - c_t$. Take the subchain of the constellation with consecutive weights $c_1, c_2, \cdots, c_{t-1}$ and add an edge with weight $c_t$ at the top. We call this new chain $K$. For $i \in \{1, \cdots, d\}$, let $Q_{K_i}$ be the point with maximal level of $K$ for which the weight going out from $Q_{K_i}$ is $i$, if this point exists. If $m = x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d}$, then the equation of $\hat{m}$ in the considered chart is

$$x_1^{f_1(n_1, \cdots, n_d) - h_1} x_2^{f_2(n_1, \cdots, n_d) - h_2} \cdots x_d^{f_d(n_1, \cdots, n_d) - h_d},$$

where $f_i$ is the left hand side of the linear equation induced by the point $Q_{K_i}$ in $\mathcal{L}(\mathcal{C})$ if $Q_{K_i}$ exists and $f_i(n_1, \cdots, n_d) = n_i$ if this point is not defined. The value $h_i$ is equal to $v_{K_i}$ if $Q_{K_i}$ exists and else $h_i = 0$.

Now suppose that $\hat{m} \cap E_i = x_1^{n'_1} x_2^{n'_2} \cdots x_d^{n'_d}$ is given. When $Q_j$ exists for every $j \in \{1, \cdots, d\}$, it follows from Proposition 3 that the linear system $\{f_j(\underline{n}) - h_j = n'_j\}_{1 \le j \le d}$ has a *unique* solution for $\underline{n}$. If not all points $Q_j$ are defined, one also sees immediately that the linear system $\{f_j(\underline{n}) - h_j = n'_j\}_{1 \le j \le d}$ is regular.

2. Let $g$ and $h$ be two functions in $\mathbb{P}F_{\underline{v}}$ with the same support $L$. Write $g = \sum_{i=1}^{s} \lambda_i m_i$ and $h = \sum_{i=1}^{s} \mu_i m_i$, with $\lambda_i$ and $\mu_i$ different from 0 $(1 \le i \le s)$. Suppose that $\{\hat{g} = 0\} \cap \mathcal{D} = \{\hat{h} = 0\} \cap \mathcal{D}$ and that $\lambda_j \neq \mu_j$. For lack of a subset $D$ of $J$ with property (2) and because of part 1 of the proof, it follows that $\lambda_i / \mu_i = \lambda_j / \mu_j$, for all $i \in J$ and so $g = h$.

■

The functions we are interested in are the functions $g$ such that $\{\hat{g} = 0\}$ does not contain any non-empty intersection $E_a \cap E_b$. They can be characterized as follows:

**Lemma 3.**— *Let $a$ and $b$ be different elements of $J$ such that $E_a \cap E_b \neq \emptyset$. Then*

$$\{\hat{g} = 0\} \textit{ contains } E_a \cap E_b$$
$$\Updownarrow$$
$$\textit{there is no } m \textit{ in supp}(g) \textit{ for which } \nu_a(m) = v_a \textit{ and } \nu_b(m) = v_b.$$

*Proof.*

Suppose that there is no $m$ in $\mathrm{supp}(g)$ for which $\nu_a(m) = v_a$ and $\nu_b(m) = v_b$. In an affine chart where one sees $E_a \cap E_b$, one has

$$\hat{g} = \hat{g_a} + \hat{g_b} + \hat{r}, \quad E_a \leftrightarrow x_a = 0, \qquad E_b \leftrightarrow x_b = 0$$

where $g_a$ is the part of $g$ with $\mathrm{supp}(g_a) = \{m \in \mathrm{supp}(g) \mid \nu_a(m) = \nu_a(g)\}$, $g_b$ is the part of $g$ with $\mathrm{supp}(g_b) = \{m \in \mathrm{supp}(g) \mid \nu_b(m) = \nu_b(g)\}$ and $r$ is $g - g_a - g_b$. From Lemma 2 it follows that $\hat{g_b} + \hat{r} \in (x_a)$ and $\hat{g_a} + \hat{r} \in (x_b)$. Then also $\hat{g} \in (x_a, x_b)$ and hence $\{\hat{g} = 0\}$ contains $E_a \cap E_b$.

When $\{\hat{g} = 0\}$ contains $E_a \cap E_b$, there exists an affine chart in which one has

$$\hat{g} = x_a g_a + x_b g_b + x_a x_b g_r, \quad E_a \leftrightarrow x_a = 0, \qquad E_b \leftrightarrow x_b = 0,$$

with $g_a \notin (x_b)$ and $g_b \notin (x_a)$. If $m \in \mathrm{supp}(g)$ such that $\nu_a(m) = v_a$ and $\nu_b(m) = v_b$, Lemma 2 implies that $\hat{m} \in \mathrm{supp}(x_a g_a) \cap \mathrm{supp}(x_b g_b)$ what is impossible.

■

Now let $g \in \mathbb{P}F_{\underline{v}}$ and let $L$ be the support of $g$. Taking the above characterization into account, we see that if $\hat{g}$ does not contain $E_a \cap E_b$, then there exists no $D \subset J$ for which $a \in D$, $b \notin D$ and $(\cup_{d\in D}\Lambda_{L,d}) \cap (\cup_{d\notin D}\Lambda_{L,d}) = \emptyset$. Note that the other implication is false in general (see for example the constellation given above with $a = 1$ and $b = 2$).
We denote

$$Z_{\underline{v},L} := \{g \in Z_{\underline{v}} \mid \mathrm{supp}(g) = L\} \text{ and } D_{\underline{v},L} := \{\{\hat{g} = 0\} \cap \mathcal{D} \in D_{\underline{v}} \mid \mathrm{supp}(g) = L\}.$$

Then we can write $Z_{\underline{v}}$ as a disjoint union $\cup_L Z_{\underline{v},L}$ where for each $L$ holds that there is no subset $D$ of $J$ satisfying condition (2). Proposition 4 tells us that the map

$$\begin{array}{rcl} \psi_L : Z_{\underline{v},L} & \longrightarrow & D_{\underline{v},L} \\ g & \longmapsto & \{\hat{g} = 0\} \cap \mathcal{D} \end{array}$$

is a bijection and then part 1 of the proof of Proposition 4 allows us to conclude that the map

$$\begin{array}{rcl} \phi : Z_{\underline{v}} & \longrightarrow & D_{\underline{v}} \\ g & \longmapsto & \{\hat{g} = 0\} \cap \mathcal{D} \end{array}$$

is bijective.

Now by Lemma 2 one can see that $\phi$ is a homeomorphism. Then we have that $\chi(Z_{\underline{v}}) = \chi(D_{\underline{v}})$ which completes the proof of Theorem 2.

■

**Corollary.**— *Let $\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_s$ be the different columns of the linear system determined by the constellation. Then*

$$P(\underline{t}) = \frac{1}{(1 - \underline{t}^{\underline{v}_1})^{\chi(D_{\underline{v}_1})} \cdots (1 - \underline{t}^{\underline{v}_s})^{\chi(D_{\underline{v}_s})}}.$$

*Proof.* When the cone $C$ associated to the linear system $\mathcal{L}(\mathcal{C})$ is non-degenerated, then one has for each $\underline{v}_i$ $(1 \le i \le s)$ that $\chi(D_{\underline{v}_i}) = N(\underline{v}_i) = 1$. If $C$ is degenerated then we have for all but one $\underline{v}_i$ $(1 \le i \le s)$ that $\chi(D_{\underline{v}_i}) = N(\underline{v}_i) = 1$. For the column $\underline{v}$ that appears more than once, one gets $\chi(D_{\underline{v}}) = N(\underline{v}) = k$, with $k = d - s + 1$.

■

As a consequence of Theorem 2 and the corollary, we obtain that the value $\chi(D_{\underline{v}})$ can be calculated from the values $\chi(D_{\underline{v}_1}), \cdots, \chi(D_{\underline{v}_s})$ and $k$.

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